

An Extension of Feller's Strong Law of Large Numbers

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Abstract This paper presents a general result that allows for establishing a link between the Kolmogorov-Marcinkiewicz-Zygmund strong law of large numbers and Feller's strong law of large numbers in a Banach space setting. Let $\{X, X_n; n \geq 1\}$ be a sequence of independent and identically distributed Banach space valued random variables and set $S_n = \sum_{i=1}^n X_i$, $n \geq 1$. Let $\{a_n; n \geq 1\}$ and $\{b_n; n \geq 1\}$ be increasing sequences of positive real numbers such that $\lim_{n \rightarrow \infty} a_n = \infty$ and $\{b_n/a_n; n \geq 1\}$ is a nondecreasing sequence. We show that

$$\frac{S_n - n\mathbb{E}(XI\{\|X\| \leq b_n\})}{b_n} \rightarrow 0 \text{ almost surely}$$

for every Banach space valued random variable X with $\sum_{n=1}^{\infty} \mathbb{P}(\|X\| > b_n) < \infty$ if $S_n/a_n \rightarrow 0$ almost surely for every symmetric Banach space valued random variable X with $\sum_{n=1}^{\infty} \mathbb{P}(\|X\| > a_n) < \infty$. To establish this result, we invoke two tools (obtained recently by Li, Liang, and Rosalsky): a symmetrization procedure for the strong law of large numbers and a probability inequality for sums of independent Banach space valued random variables.

Keywords Feller's strong law of large numbers · Kolmogorov-Marcinkiewicz-Zygmund strong law of large numbers · Rademacher type p Banach space · Sums of independent random variables

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1 Introduction and the main result

We begin with stating Feller's (1946) strong law of large numbers (SLLN) as follows.

Theorem A. (Feller's SLLN. Theorems 1 and 2 of Feller (1946)). *Let $\{X, X_n; n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) real-valued random variables, and let $S_n = \sum_{i=1}^n X_i$, $n \geq 1$. Let $\{b_n; n \geq 1\}$ be an increasing sequence of positive real numbers. Suppose that one of the following two sets of conditions holds:*

(i) *For some $0 < \delta < 1$, $\mathbb{E}|X|^{1+\delta} = \infty$, $\mathbb{E}X = 0$, and there exists an ϵ with $0 \leq \epsilon < 1$ such that*

$$b_n n^{-1/(1+\epsilon)} \uparrow \text{ and } b_n/n \downarrow,$$

(ii) *$\mathbb{E}|X| = \infty$ and*

$$b_n/n \uparrow.$$

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Then we have

$$\lim_{n \rightarrow \infty} \frac{S_n}{b_n} = 0 \quad \text{almost surely (a.s.) or} \quad \limsup_{n \rightarrow \infty} \frac{|S_n|}{b_n} = \infty \quad \text{a.s.}$$

according as

$$\sum_{n=1}^{\infty} \mathbb{P}(|X| > b_n) < \infty \quad \text{or} \quad = \infty.$$

Feller's SLLN is a remarkable limit theorem concerning sums of i.i.d. random variables. From the internet, one can find that Feller's SLLN has received more than 150 citations where many of them have been received within the most recent 5 years.

This paper presents a general result in a Banach space setting that allows for establishing a link between the Kolmogorov-Marcinkiewicz-Zygmund SLLN and Feller's SLLN.

For stating our main result, we introduce some notation as follows. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\mathbf{B}, \|\cdot\|)$ be a real separable Banach space equipped with its Borel σ -algebra \mathcal{B} (= the σ -algebra generated by the class of open subsets of \mathbf{B} determined by $\|\cdot\|$). A \mathbf{B} -valued random variable X is defined as a measurable function from (Ω, \mathcal{F}) into $(\mathbf{B}, \mathcal{B})$. Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. \mathbf{B} -valued random variables and put $S_n = \sum_{i=1}^n X_i$, $n \geq 1$. Let $\{R, R_n; n \geq 1\}$ be a *Rademacher sequence*; that is, $\{R_n; n \geq 1\}$ is a sequence of i.i.d. random variables with $\mathbb{P}(R = 1) = \mathbb{P}(R = -1) = 1/2$. Let $\mathbf{B}^\infty = \mathbf{B} \times \mathbf{B} \times \mathbf{B} \times \cdots$ and define

$$\mathcal{C}(\mathbf{B}) = \left\{ (v_1, v_2, \dots) \in \mathbf{B}^\infty : \sum_{n=1}^{\infty} R_n v_n \text{ converges in probability} \right\}.$$

Let $1 \leq p \leq 2$. Then \mathbf{B} is said to be of *Rademacher type p* if there exists a constant $0 < C < \infty$ such that

$$\mathbb{E} \left\| \sum_{n=1}^{\infty} R_n v_n \right\|^p \leq C \sum_{n=1}^{\infty} \|v_n\|^p \quad \text{for all } (v_1, v_2, \dots) \in \mathcal{C}(\mathbf{B}).$$

The following remarkable theorem, which is due to de Acosta (1981), provides a characterization of Rademacher type p Banach spaces.

Theorem B. (de Acosta (1981)). *Let $1 \leq p < 2$. Then the following two statements are equivalent:*

- (i) *The Banach space \mathbf{B} is of Rademacher type p .*
- (ii) *For every sequence $\{X, X_n; n \geq 1\}$ of i.i.d. \mathbf{B} -valued variables,*

$$\lim_{n \rightarrow \infty} \frac{S_n}{n^{1/p}} = 0 \quad \text{a.s. if and only if } \mathbb{E}\|X\|^p < \infty \quad \text{and} \quad \mathbb{E}X = 0.$$

The main result of this paper is the following theorem.

Theorem 1.1. *Let $(\mathbf{B}, \|\cdot\|)$ be a real separable Banach space. Let $\{a_n; n \geq 1\}$ and $\{b_n; n \geq 1\}$ be increasing sequences of positive real numbers such that*

$$\lim_{n \rightarrow \infty} a_n = \infty \quad \text{and} \quad b_n/a_n \uparrow. \tag{1.1}$$

Suppose that for every symmetric sequence $\{X, X_n; n \geq 1\}$ of i.i.d. \mathbf{B} -valued random variables,

$$\lim_{n \rightarrow \infty} \frac{S_n}{a_n} = 0 \quad \text{a.s. whenever} \quad \sum_{n=1}^{\infty} \mathbb{P}(\|X\| > a_n) < \infty. \tag{1.2}$$

Then, for every sequence $\{X, X_n; n \geq 1\}$ of i.i.d. \mathbf{B} -valued random variables, we have that

$$\lim_{n \rightarrow \infty} \frac{S_n - \gamma_n}{b_n} = 0 \quad \text{a.s. or} \quad \limsup_{n \rightarrow \infty} \frac{\|S_n - \gamma_n\|}{b_n} = \infty \quad \text{a.s.} \quad (1.3)$$

according as

$$\sum_{n=1}^{\infty} \mathbb{P}(\|X\| > b_n) < \infty \quad \text{or} \quad = \infty. \quad (1.4)$$

Here and below $\gamma_n = n\mathbb{E}(XI\{\|X\| \leq b_n\})$, $n \geq 1$.

Remark 1.1. We now can see how Feller's SLLN can be easily derived from the Kolmogorov-Marcinkiewicz-Zygmund SLLN and Theorem 1.1 above. For given ϵ with $0 \leq \epsilon < 1$, write $a_n = n^{1/(1+\epsilon)}$, $n \geq 1$. The celebrated Kolmogorov-Marcinkiewicz-Zygmund SLLN (see Kolmogoroff (1930) for $\epsilon = 0$ and Marcinkiewicz and Zygmund (1937) for $0 < \epsilon < 1$) asserts that, for every symmetric sequence $\{X, X_n; n \geq 1\}$ of i.i.d. real-valued (i.e., $\mathbf{B} = \mathbb{R}$) random variables

$$\lim_{n \rightarrow \infty} \frac{S_n}{a_n} = \lim_{n \rightarrow \infty} \frac{S_n}{n^{1/(1+\epsilon)}} = 0 \quad \text{a.s. if and only if} \quad \mathbb{E}|X|^{1+\epsilon} < \infty \quad (\text{i.e., } \sum_{n=1}^{\infty} \mathbb{P}(|X| > a_n) < \infty).$$

Then by Theorem 1.1, for every sequence $\{X, X_n; n \geq 1\}$ of i.i.d. real-valued random variables and every increasing sequence $\{b_n; n \geq 1\}$ of positive real numbers with

$$b_n/a_n = b_n n^{-1/(1+\epsilon)} \uparrow,$$

we have

$$\lim_{n \rightarrow \infty} \frac{S_n - \gamma_n}{b_n} = 0 \quad \text{a.s. or} \quad \limsup_{n \rightarrow \infty} \frac{|S_n - \gamma_n|}{b_n} = \infty \quad \text{a.s.}$$

according as

$$\sum_{n=1}^{\infty} \mathbb{P}(|X| > b_n) < \infty \quad \text{or} \quad = \infty,$$

where $\gamma_n = n\mathbb{E}(XI\{|X| \leq b_n\})$, $n \geq 1$. That is, Feller's SLLN follows from the Kolmogorov-Marcinkiewicz-Zygmund SLLN and Theorem 1.1 above.

Remark 1.2. Under the assumptions of Theorem 1.1, it follows from the conclusion of Theorem 1.1 that, for every sequence $\{X, X_n; n \geq 1\}$ of i.i.d. \mathbf{B} -valued random variables,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{S_n - \gamma_n}{b_n} = 0 \quad \text{a.s. if and only if} \quad \sum_{n=1}^{\infty} \mathbb{P}(\|X\| > b_n) < \infty, \\ \limsup_{n \rightarrow \infty} \frac{\|S_n - \gamma_n\|}{b_n} = \infty \quad \text{a.s. if and only if} \quad \sum_{n=1}^{\infty} \mathbb{P}(\|X\| > b_n) = \infty. \end{aligned}$$

Hence under the assumptions of Theorem 1.1, there does not exist a sequence $\{X, X_n; n \geq 1\}$ of i.i.d. \mathbf{B} -valued random variables such that

$$0 < \limsup_{n \rightarrow \infty} \frac{\|S_n - \gamma_n\|}{b_n} < \infty \quad \text{a.s.}$$

Also, combining Theorem 1.1 and Theorem B above, we immediately obtain the following two remarks.

Remark 1.3. Let $1 \leq p < 2$ and let $\{a_n; n \geq 1\}$ be an increasing sequences of positive real numbers such that

$$\lim_{n \rightarrow \infty} a_n = \infty \text{ and } n^{1/p}/a_n \uparrow.$$

Let $(\mathbf{B}, \|\cdot\|)$ be a real separable Banach space such that (1.2) holds for every symmetric sequence $\{X, X_n; n \geq 1\}$ of i.i.d. \mathbf{B} -valued random variables. Then the Banach space \mathbf{B} is of Rademacher type p .

Remark 1.4. Let $1 \leq p < 2$ and let $\{b_n; n \geq 1\}$ be a sequence of positive real numbers such that

$$b_n/n^{1/p} \uparrow.$$

If \mathbf{B} is of Rademacher type p , then for every sequence $\{X, X_n; n \geq 1\}$ of i.i.d. \mathbf{B} -valued random variables, (1.3) and (1.4) are equivalent.

Remark 1.5. Remark 1.4 should be compared with Theorem 4 of Adler, Rosalsky, and Taylor (1989) and with (in the unweighted case) the key lemma (Lemma 6) of that article. In Lemma 6 (wherein $1 \leq p \leq 2$) and Theorem 4 (wherein $1 < p \leq 2$) of Adler, Rosalsky, and Taylor (1989), for a sequence $\{X, X_n; n \geq 1\}$ of i.i.d. random variables in a real separable Rademacher type p Banach space and a sequence of positive constants $b_n \uparrow \infty$, conditions are provided under which

$$\sum_{n=1}^{\infty} \mathbb{P}(\|X\| > b_n) < \infty$$

ensures that

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n (X_i - \mathbb{E}(X_i \{ \|X\| \leq b_i \}))}{b_n} = 0 \text{ a.s.}$$

and

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n X_i}{b_n} = 0 \text{ a.s.}$$

The proof of Theorem 1.1 will be provided in Section 2. To establish Theorem 1.1, we invoke two tools (obtained recently by Li, Liang, and Rosalsky (2017a, b)): a symmetrization procedure for the SLLN and a probability inequality which is a comparison theorem for sums of independent \mathbf{B} -valued random variables.

We close this section by remarking that a version of Feller's SLLN was obtained by Martikainen and Petrov (1980) for a sequence of identically distributed real-valued random variables $\{X, X_n; n \geq 1\}$ without any independence conditions being imposed on the summands; the result holds irrespective of the joint distributions of the summands. Specifically, in Theorem 2 of Martikainen and Petrov (1980) it is shown that

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n X_i}{b_n} = 0 \text{ a.s.}$$

if

$$0 < b_n \uparrow, \quad b_n \sum_{i=n}^{\infty} \frac{1}{b_i} = O(n), \quad \text{and} \quad \sum_{n=1}^{\infty} \mathbb{P}(|X| > b_n) < \infty.$$

2 Proof of Theorem 1.1

Throughout this section, $\{a_n; n \geq 1\}$ and $\{b_n; n \geq 1\}$ are increasing sequences of positive real numbers satisfying (1.1). Write

$$I(1) = \{i; b_i \leq 2\} \quad \text{and} \quad I(m) = \{i; 2^{m-1} < b_i \leq 2^m\}, \quad m \geq 2.$$

It follows from (1.1) that $0 < b_n \uparrow \infty$ and

$$\{1, 2, 3, \dots, n, \dots\} = \bigcup_{m=1}^{\infty} I(m).$$

Note that the $I(m)$, $m \geq 1$ are mutually exclusive sets. Thus there exist positive integers k_n, m_n , $n \geq 1$ such that

$$k_1 < k_2 < \dots < k_n < \dots, \quad m_1 < m_2 < \dots < m_n < \dots,$$

$$\{1, 2, 3, \dots, n, \dots\} = \bigcup_{n=1}^{\infty} I(m_n), \quad \text{and} \quad I(m_1) = \{1, \dots, k_1\}, \quad I(m_n) = \{k_{n-1} + 1, \dots, k_n\}, \quad n \geq 2.$$

To prove Theorem 1.1, we use the following four preliminary lemmas.

Lemma 2.1. (Lemma 3.1 of Li, Liang, and Rosalsky (2017 b)) *There exist two continuous and increasing functions $\varphi(\cdot)$ and $\psi(\cdot)$ defined on $[0, \infty)$ such that*

$$\lim_{t \rightarrow \infty} \varphi(t) = \infty \quad \text{and} \quad \frac{\psi(\cdot)}{\varphi(\cdot)} \text{ is a nondecreasing function on } [0, \infty), \quad (2.1)$$

$$\varphi(0) = \psi(0) = 0, \quad \varphi(n) = a_n, \quad \psi(n) = b_n, \quad n \geq 1. \quad (2.2)$$

Lemma 2.2. *Let $\{V_n; n \geq 1\}$ be a sequence of independent and symmetric \mathbf{B} -valued random variables. Set $k_0 = 0$. Then the following two statements hold.*

(i) *If*

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n V_i}{a_n} = 0 \quad \text{a.s.}, \quad (2.3)$$

then

$$\sum_{n=1}^{\infty} \mathbb{P} \left(\left\| \sum_{i=k_{n-1}+1}^{k_n} V_i \right\| > \epsilon a_{k_n} \right) < \infty \quad \forall \epsilon > 0. \quad (2.4)$$

(ii)

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n V_i}{b_n} = 0 \quad \text{a.s.} \quad (2.5)$$

if and only if

$$\sum_{n=1}^{\infty} \mathbb{P} \left(\left\| \sum_{i=k_{n-1}+1}^{k_n} V_i \right\| > \epsilon b_{k_n} \right) < \infty \quad \forall \epsilon > 0. \quad (2.6)$$

Proof We first prove Part (i). Clearly, (2.3) implies that

$$\frac{\left\| \sum_{i=k_{n-1}+1}^{k_n} V_i \right\|}{a_{k_n}} \leq \frac{\left\| \sum_{i=1}^{k_n} V_i \right\|}{a_{k_n}} + \left(\frac{a_{k_{n-1}}}{a_{k_n}} \right) \frac{\left\| \sum_{i=1}^{k_{n-1}} V_i \right\|}{a_{k_{n-1}}} \rightarrow 0 \text{ a.s. as } n \rightarrow \infty. \quad (2.7)$$

Since the $\sum_{i=k_{n-1}+1}^{k_n} V_i$, $n \geq 1$ are independent \mathbf{B} -valued random variables, (2.4) follows from (2.7) and the Borel-Cantelli lemma.

We now establish Part (ii). From the proof of Part (i), we only need to show that (2.5) follows from (2.6). Since $\{V_n; n \geq 1\}$ is a sequence of independent and symmetric \mathbf{B} -valued random variables, by the remarkable Lévy inequality in a Banach space setting (see, e.g., see Proposition 2.3 of Ledoux and Talagrand (1991)), we have that for every $n \geq 1$,

$$\mathbb{P} \left(\max_{k_{n-1} < k \leq k_n} \left\| \sum_{i=k_{n-1}+1}^k V_i \right\| > \epsilon b_{k_n} \right) \leq 2\mathbb{P} \left(\left\| \sum_{i=k_{n-1}+1}^{k_n} V_i \right\| > \epsilon b_{k_n} \right) \quad \forall \epsilon > 0.$$

Thus it follows from (2.6) that

$$\sum_{n=1}^{\infty} \mathbb{P} \left(\max_{k_{n-1} < k \leq k_n} \left\| \sum_{i=k_{n-1}+1}^k V_i \right\| > \epsilon b_{k_n} \right) < \infty \quad \forall \epsilon > 0$$

which ensures that

$$A_n \triangleq \frac{\max_{k_{n-1} < k \leq k_n} \left\| \sum_{i=k_{n-1}+1}^k V_i \right\|}{b_{k_n}} \rightarrow 0 \text{ a.s.} \quad (2.8)$$

Now by the Toeplitz lemma, we conclude from (2.8) that

$$\begin{aligned} \max_{k_{n-1} < k \leq k_n} \frac{\left\| \sum_{i=1}^k V_i \right\|}{b_k} &\leq 2 \max_{k_{n-1} < k \leq k_n} \frac{\left\| \sum_{i=1}^k V_i \right\|}{b_{k_n}} \\ &\leq 2 \left(\sum_{j=1}^{n-1} \frac{\left\| \sum_{i=k_{j-1}+1}^{k_j} V_i \right\|}{b_{k_n}} + A_n \right) \\ &\leq 2 \sum_{j=1}^n \left(\frac{b_{k_j}}{b_{k_n}} \right) A_j \\ &\leq 4 \sum_{j=1}^n \left(\frac{2^{m_j}}{2^{m_n}} \right) A_j \\ &\rightarrow 0 \text{ a.s. as } n \rightarrow \infty; \end{aligned}$$

i.e., (2.5) holds. \square

The following probability inequality is due to Li, Liang, and Rosalsky (2017 b) and is a comparison theorem for sums of independent \mathbf{B} -valued random variables.

Lemma 2.3. (Theorem 1.1 (ii) of Li, Liang, and Rosalsky (2017 b)) *Let $\varphi(\cdot)$ and $\psi(\cdot)$ be two continuous and increasing functions defined on $[0, \infty)$ with $\varphi(0) = \psi(0) = 0$ and satisfying (2.1). If $\{V_n; n \geq 1\}$ is a sequence of independent and symmetric \mathbf{B} -valued random variables, then for every $n \geq 1$ and all $t \geq 0$,*

$$\mathbb{P} \left(\left\| \sum_{i=1}^n V_i \right\| > tb_n \right) \leq 4\mathbb{P} \left(\left\| \sum_{i=1}^n \varphi(\psi^{-1}(\|V_i\|)) \frac{V_i}{\|V_i\|} \right\| > ta_n \right) + \sum_{i=1}^n \mathbb{P}(\|V_i\| > b_n).$$

The following symmetrization procedure for the SLLN for independent \mathbf{B} -valued random variables is due to Li, Liang, and Rosalsky (2017 a).

Lemma 2.4. (Corollary 1.3 of Li, Liang, and Rosalsky (2017 a)) *Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. \mathbf{B} -valued random variables. Let $\{X'_n; n \geq 1\}$ be an independent copy of $\{X_n; n \geq 1\}$. Write $S_n = \sum_{i=1}^n X_i$, $S'_n = \sum_{i=1}^n X'_i$, $n \geq 1$. Let $\{b_n; n \geq 1\}$ be an increasing sequence of positive real numbers such that $\lim_{n \rightarrow \infty} b_n = \infty$. Then*

$$\lim_{n \rightarrow \infty} \frac{S_n - n\mathbb{E}(XI\{\|X\| \leq b_n\})}{b_n} = 0 \quad \text{a.s.}$$

if and only if

$$\lim_{n \rightarrow \infty} \frac{S_n - S'_n}{b_n} = 0 \quad \text{a.s.}$$

With the preliminaries accounted for, Theorem 1.1 may be proved.

Proof of Theorem 1.1 To establish this theorem, it suffices to show that, for every sequence $\{X, X_n; n \geq 1\}$ of i.i.d. \mathbf{B} -valued random variables, the following three statements are equivalent:

$$\lim_{n \rightarrow \infty} \frac{S_n - \gamma_n}{b_n} = 0 \quad \text{a.s.}, \quad (2.9)$$

$$\limsup_{n \rightarrow \infty} \frac{\|S_n - \gamma_n\|}{b_n} < \infty \quad \text{a.s.}, \quad (2.10)$$

$$\sum_{n=1}^{\infty} \mathbb{P}(\|X\| > b_n) < \infty. \quad (2.11)$$

The three statements (2.9)-(2.11) are equivalent if we can show that (2.9) and (2.11) are equivalent and (2.9) and (2.10) are equivalent.

For establishing the implication “(2.9) \Rightarrow (2.11)”, let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. \mathbf{B} -valued random variables satisfying (2.9). It follows from (2.9) that

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n (X_i - X'_i)}{b_n} = 0 \quad \text{a.s.}$$

which implies that

$$\frac{X_n - X'_n}{b_n} \rightarrow 0 \quad \text{a.s.} \quad (2.12)$$

By the Borel-Cantelli lemma, (2.12) is equivalent to

$$\sum_{n=1}^{\infty} \mathbb{P}(\|X - X'\| > \epsilon b_n) = \sum_{n=1}^{\infty} \mathbb{P}(\|X_n - X'_n\| > \epsilon b_n) < \infty \quad \forall \epsilon > 0. \quad (2.13)$$

Note that $\{\|X'\| \leq b_n/2, \|X\| > b_n\} \subseteq \{\|X - X'\| > b_n/2\}$ and $\lim_{n \rightarrow \infty} \mathbb{P}(\|X'\| \leq b_n/2) = 1$. We thus have that, for all large n ,

$$\mathbb{P}(\|X\| > b_n) \leq 2\mathbb{P}(\|X - X'\| > b_n/2)$$

which, together with (2.13), implies (2.11).

We now prove “(2.11) \Rightarrow (2.9)”. Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. \mathbf{B} -valued random variables satisfying (2.11). Since $\{a_n; n \geq 1\}$ and $\{b_n; n \geq 1\}$ are increasing sequences of positive real numbers satisfying (1.1), by Lemma 2.1, there exist two continuous and increasing functions $\varphi(\cdot)$ and $\psi(\cdot)$ defined on $[0, \infty)$ such that both (2.1) and (2.2) hold. Write

$$\tilde{X} = \frac{X - X'}{2}, \quad \tilde{X}_n = \frac{X_n - X'_n}{2}, \quad n \geq 1$$

and

$$Y = \varphi\left(\psi^{-1}(\|\tilde{X}\|)\right) \frac{\tilde{X}}{\|\tilde{X}\|}, \quad Y_n = \varphi\left(\psi^{-1}(\|\tilde{X}_n\|)\right) \frac{\tilde{X}_n}{\|\tilde{X}_n\|}, \quad n \geq 1.$$

Then $\{\tilde{X}, \tilde{X}_n; n \geq 1\}$ is a sequence of i.i.d. symmetric \mathbf{B} -valued random variables such that

$$\sum_{n=1}^{\infty} \mathbb{P}(\|\tilde{X}\| > b_n) \leq 2 \sum_{n=1}^{\infty} \mathbb{P}(\|X\| > b_n) < \infty \quad (2.14)$$

and $\{Y, Y_n; n \geq 1\}$ is a sequence of i.i.d. symmetric \mathbf{B} -valued random variables such that

$$\sum_{n=1}^{\infty} \mathbb{P}(\|Y\| > a_n) = \sum_{n=1}^{\infty} \mathbb{P}(\|\tilde{X}\| > b_n) < \infty. \quad (2.15)$$

Hence, by Lemma 2.2 (i), we conclude from (2.15) and (1.2) that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left\|\sum_{i=k_{n-1}+1}^{k_n} Y_i\right\| > \epsilon a_{k_n}\right) < \infty \quad \forall \epsilon > 0. \quad (2.16)$$

By Lemma 2.3, we have that, for every $n \geq 1$,

$$\begin{aligned} \mathbb{P}\left(\left\|\sum_{i=k_{n-1}+1}^{k_n} \tilde{X}_i\right\| > \epsilon b_{k_n}\right) &\leq 4\mathbb{P}\left(\left\|\sum_{i=k_{n-1}+1}^{k_n} Y_i\right\| > \epsilon a_n\right) + \sum_{i=k_{n-1}+1}^{k_n} \mathbb{P}(\|\tilde{X}_i\| > b_{k_n}) \\ &\leq 4\mathbb{P}\left(\left\|\sum_{i=k_{n-1}+1}^{k_n} Y_i\right\| > \epsilon a_n\right) + \sum_{i=k_{n-1}+1}^{k_n} \mathbb{P}(\|\tilde{X}\| > b_i) \quad \forall \epsilon > 0. \end{aligned}$$

It thus follows from (2.15) and (2.16) that

$$\begin{aligned}
& \sum_{n=1}^{\infty} \mathbb{P} \left(\left\| \sum_{i=k_{n-1}+1}^{k_n} \tilde{X}_i \right\| > \epsilon b_{k_n} \right) \\
& \leq 4 \sum_{n=1}^{\infty} \mathbb{P} \left(\left\| \sum_{i=k_{n-1}+1}^{k_n} Y_i \right\| > \epsilon a_n \right) + \sum_{n=1}^{\infty} \sum_{i=k_{n-1}+1}^{k_n} \mathbb{P} \left(\|\tilde{X}_i\| > b_i \right) \\
& = 4 \sum_{n=1}^{\infty} \mathbb{P} \left(\left\| \sum_{i=k_{n-1}+1}^{k_n} Y_i \right\| > \epsilon a_n \right) + \sum_{n=1}^{\infty} \mathbb{P} \left(\|\tilde{X}_i\| > b_n \right) \\
& < \infty \quad \forall \epsilon > 0.
\end{aligned} \tag{2.17}$$

By Lemma 2.2 (ii), (2.17) is equivalent to

$$\frac{S_n - S'_n}{2b_n} = \frac{\sum_{i=1}^n \tilde{X}_i}{b_n} \rightarrow 0 \quad \text{a.s.}$$

Hence

$$\frac{S_n - S'_n}{b_n} \rightarrow 0 \quad \text{a.s.}$$

By Lemma 2.4, (2.9) follows.

The implication “(2.9) \Rightarrow (2.10)” is obvious.

We now establish the implication “(2.10) \Rightarrow (2.9)”. It follows from (2.10) that

$$\limsup_{n \rightarrow \infty} \frac{\|\sum_{i=1}^n (X_i - X'_i)\|}{b_n} < \infty \quad \text{a.s.}$$

which implies that

$$\limsup_{n \rightarrow \infty} \frac{\|X_n - X'_n\|}{b_n} < \infty \quad \text{a.s.} \tag{2.18}$$

By the Borel-Cantelli lemma, (2.18) is equivalent to: for some constant $0 < \lambda < \infty$,

$$\sum_{n=1}^{\infty} \mathbb{P} (\|X_n - X'_n\| > \lambda b_n) < \infty; \quad \text{i.e.,} \quad \sum_{n=1}^{\infty} \mathbb{P} \left(\left\| \frac{X - X'}{\lambda} \right\| > b_n \right) < \infty.$$

That is, (2.11) holds with X replaced by symmetric random variable $(X - X')/\lambda$. Since (2.9) and (2.11) are equivalent, we conclude that

$$\left(\frac{1}{\lambda} \right) \frac{S_n - S'_n}{b_n} = \frac{\sum_{i=1}^n \frac{X_i - X'_i}{\lambda}}{b_n} \rightarrow 0 \quad \text{a.s.}$$

Thus

$$\frac{S_n - S'_n}{b_n} \rightarrow 0 \quad \text{a.s.}$$

which, by Lemma 2.4, implies (2.9). The proof of Theorem 1.1 is therefore complete. \square

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